A Construction of the Null Set

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Abstract

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1 Motivation

The imaginary number i satisfies the relation $i^2 + 1 = 0$, but this specification falls short of a construction. Accordingly, one performs a construction by extending the reals to the the quotient of the ideal $(X^2 + 1)$ in $\mathbb{R}[X]$, the ring of real polynomials in X, and showing that the element $X + (X^2 + 1)$ has all of the desired algebraic properties of i. This procedure fully constructs the complex numbers as an extension of the reals. Perhaps the empty set could also be explicitly constructed with the appropriate extension? This article explores this question.

2 A Construction

We propose the following construction: Let U be any set with at least one member. We begin with the collection

$$V = \{(T_1, T_2); T_1 \subseteq U, T_2 \subseteq U, T_1 \text{ has at least one member, } T_2 \text{ has at least one member.}\}$$

and define an equivalence relation on members of V. Indeed, we say that for $(A, B), (C, D) \in V$,

$$(A, B) \equiv (C, D)$$

if and only if $x \in A$ and $x \in B \Leftrightarrow x \in C$ and $x \in D$. To put it another way, either A and B have no members in common and neither do C and D, or A and B have members in common, C and D have members in common, and $A \cap B = C \cap D$.

We now observe that we can embed injectively every subset S containing at least one member into the collection of equivalence classes as follows: Let $\overline{(E, F)}$ denote the equivalence class containing the pair (E, F). For sets $S \subseteq U$ that have at least one member,, we define the mapping f according to $f : S \mapsto \overline{(S, S)}$. This mapping is injective on the collection of sets with at least one member; indeed, suppose $f(S_1) = f(S_2)$, then $\overline{(S_1, S_1)} = \overline{(S_2, S_2)}$, thus $(S_1, S_1) \equiv (S_2, S_2)$, which gives $S_1 \cap S_1 = S_2 \cap S_2$, i.e. $S_1 = S_2$. The image of f does not include the equivalence class of pairs of sets with no elements in common; we define this as the empty set, and denote it by $\emptyset$.

We now define set operations on the collection of equivalence classes as follows:

$\overline{(A, A)} \cup \overline{(B, B)} \equiv \overline{(A \cup B, A \cup B)}$

For A, B with members in common, $\overline{(A, A)} \cap \overline{(B, B)} \equiv \overline{(A \cap B, A \cap B)}$.

For any proper subset A of U with at least one member, $\overline{(A, A)}^c \equiv \overline{(A^c, A^c)}$

For A, B with no members in common, $\overline{(A, A)} \cap \overline{(B, B)} \equiv \emptyset$.

$\overline{(U, U)}^c \equiv \emptyset$

As defined, the mapping f is a set homomorphism.

3 REFERENCES

[1] TBS Dunford, Nelson & Schwartz, Jacob T. (1988), Linear Operators, General Theory, Wiley-Interscience